Computational Materials Physics

Center for Molecular Modeling

Department of Materials Science and Engineering

proof of the first Hohenberg-Kohn theorem

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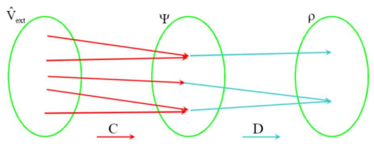
http://molmod.ugent.be
http://www.ugent.be/ea/dmse/en
my talks on Youtube: http://goo.gl/P2b1Hs

1st Hohenberg-Kohn theorem



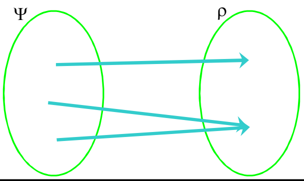
$C : \hat{V}_{ext} \mapsto \Psi$ surjective by construction

$D : \Psi \mapsto \rho$ surjective by construction

Are C and D also injective ?

for non-degenerate ground states

To proof: if $\Psi \neq \Psi'$, then $\rho \neq \rho'$
Do this by assuming that $\Psi \neq \Psi'$ and $\rho = \rho'$,
which will lead to a contradiction.



Ψ is the ground state of a hamiltonian H, ground state energy is E_{gs}

Ψ' is the ground state of a hamiltonian H', ground state energy is E'_{gs}

Same number of electrons (otherwise trivial), hence:

H = T + W +V and H' = T + W + V'

➔ H = H' – V' + V

Due to variational principle:

$$E_{gs} = \langle \Psi | H | \Psi \rangle < \langle \Psi' | H | \Psi' \rangle$$

$$= \langle \Psi' | H' + V - V' | \Psi' \rangle$$

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$$= E'_{gs} + \int \rho'(\bar{r}) \left[v(\bar{r}) - v'(\bar{r}) \right] d\bar{r}$$

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$$E_{gs} < E'_{gs} + \int \rho'(\bar{r}) \left[v(\bar{r}) - v'(\bar{r}) \right] d\bar{r}$$

Repeat argument, starting with Ψ' :

$$E'_{gs} < E_{gs} + \int \rho(\bar{r}) \left[v'(\bar{r}) - v(\bar{r}) \right] d\bar{r}$$

$$E_{gs} < E'_{gs} + \int \rho'(\bar{r})\left[v(\bar{r}) - v'(\bar{r})\right]d\bar{r}$$

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Add, assuming that $\rho = \rho'$

➔ $E_{gs} + E'_{gs} < E_{gs} + E'_{gs}$

➔ contradiction

for non-degenerate ground states

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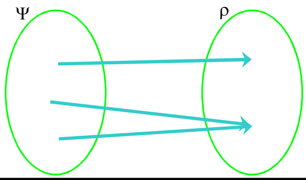
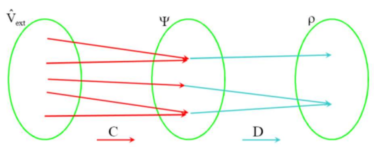


1st Hohenberg-Kohn theorem



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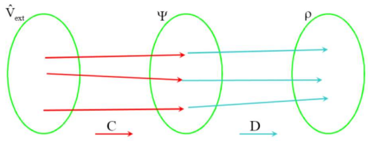
1^{st} Hohenberg-Kohn theorem

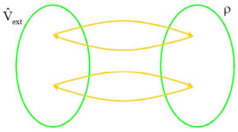


$C : \hat{V}_{ext} \mapsto \Psi$ injective (no surprise)

$D : \Psi \mapsto \rho$ injective (very surprizing !)
= first theorem of Hohenberg and Kohn (1964)

1^{st} Hohenberg-Kohn theorem

" There is a one-to-one correspondence between the ground state density of a many-electron system and the external potential. "



for non-degenerate ground states

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Repeat argument, starting with Ψ' :

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Add, assuming that $\rho = \rho'$ ➔ $E_{gs} + E'_{gs} < E_{gs} + E'_{gs}$ ➔ contradiction

P. Hohenberg and W. Kohn, Phys. Rev. 136B (1964) 864